

Compressibility regularizes the $\mu(I)$ -rheology for dense granular flows

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(Received xx; revised xx; accepted xx)

The $\mu(I)$ -rheology was recently proposed as a potential candidate to model the incompressible flow of frictional grains in the dense inertial regime. However, this rheology was shown to be ill-posed in the mathematical sense for a large range of parameters, notably in the low and large inertial number limits (Barker *et al.* 2015). In this rapid communication, we extend the stability analysis of Barker *et al.* (2015) to compressible flows. We show that compressibility regularizes mostly the equations, making the problem well-posed for all parameters, at the condition that sufficient dissipation be associated with volume changes. In addition to the usual Coulomb shear friction coefficient μ , we introduce a bulk friction coefficient μ_b , associated to volume changes and show that the problem is well-posed in two dimensions if $\mu_b > 2-2\mu$ ($\mu_b > 3-7\mu/2$ in three dimensions). Moreover, we show that the ill-posed domain defined by Barker *et al.* (2015) transforms into a domain where the flow is unstable but remain well-posed when compressibility is taken into account. These results suggest the importance of taking into account dynamic compressibility for the modelling of dense granular flows and open new perspectives to investigate the emission and propagation of acoustic waves inside these flows.

Key words:

1. Introduction

The so called $\mu(I)$ -rheology was recently proposed to model granular flows in the dense inertial regime (GDR MiDi 2004; da Cruz *et al.* 2005). This rheology rests on the fact that unidirectional granular shear flows are fairly well described using a single friction coefficient μ —ratio of the shear stress τ to the confinement pressure p —that varies with an inertial (dimensionless) number defined as $I = d\dot{\gamma}/\sqrt{p/\rho}$, where d is the grain diameter, $\dot{\gamma}$ is the shear rate, ρ is the granular flow density. I can be interpreted as the ratio of a microscopic grain rearrangement time scale to a macroscopic flow time scale. This rheology may be thought as a generalization of the basic Coulomb friction model $\tau/p = \mu$, with a friction coefficient that varies according to the local shear rate and confinement pressure.

This simple scaling was shown to break at low inertial numbers close to the jamming limit, where non-local effects become important (Kamrin & Koval 2012). On the other hand, at very high inertial numbers, granular flows enter the collisional regime and are best described by kinetic theory. In the limit of incompressible flows, the $\mu(I)$ -rheology was also recently shown to be ill-posed for a large range of parameters, notably for high

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and low inertial numbers (Barker *et al.* 2015). For this range of parameters, infinitely small wavelengths are indeed amplified at an unbounded rate, which yields non-physical solutions. In comparison, Schaeffer (1987) and Pitman & Schaeffer (1987) showed that the constant Coulomb friction rheology is always ill-posed in the incompressible limit but that adding compressibility effects would “greatly regularize the equations”.

Granular flows are indeed prone to dilate or contract in response to deformation. In the collisional limit, the volume occupied by granular flows depends directly on the granular temperature, which itself is a function of the imposed shear, allowing thus the propagation of acoustic waves. Forterre & Pouliquen (2002) showed that kinetic theory taken in the compressible limit could reproduce the instability leading to the longitudinal vortices observed in their experiments. Dense granular flows in the inertial regime might also be strongly affected by acoustic waves, resonances and instabilities due to dynamic density fluctuations as recently observed in numerical and experimental studies (Börzsönyi *et al.* 2009; Brodu *et al.* 2013; Trulsson *et al.* 2013; Krishnaraj & Nott 2016). A classical example of such instability is also found in the pulsating flows frequently observed out of silos.

In this rapid communication, we extend Barker *et al.* (2015) analysis and show that taking into account the weak compressibility of granular flows regularizes mostly the $\mu(I)$ -rheology. We found that the problem is always well-posed providing that the energy dissipation due to volume change is sufficiently important. In the limit of incompressible flows, we recover the ill-posed criteria given by (Barker *et al.* 2015). When compressibility is taken into account, the flow becomes linearly unstable (so that flow structures may develop) but the problem remains well-posed.

The demonstration is organized as follows. We start by recalling the equations pertaining to the $\mu(I)$ -rheology and generalize them for compressible flows. We then consider the simple case of a plane shear flow and probe the stability of the equations in the limit of perturbations of infinitely small wavelengths. Solving the dispersion equation allows us to demonstrate the general well-posed behaviour of the system.

2. A compressible rheology

The $\mu(I)$ -rheology assumes a dependence of the effective friction coefficient and the volume fraction upon the inertial number. Jop *et al.* (2006) generalized this experimental scaling to a tensorial rheology for incompressible flows, which, within the notations of Barker *et al.* (2015), reads

$$\frac{\boldsymbol{\tau}}{p} = \mu(I) \frac{\mathbf{D}}{\|\mathbf{D}\|}, \quad I = \frac{2d\|\mathbf{D}\|}{\sqrt{p/\rho}} \quad (2.1)$$

where $\boldsymbol{\tau}$ is the deviatoric part of the stress tensor, $\mathbf{D} = (\partial_j u_i + \partial_i u_j)/2$ is the strain rate tensor and $\|\mathbf{D}\| = (\text{tr}(\mathbf{D}^2)/2)^{1/2}$ is the second tensorial invariant. This specific form arises from the association of a Von Mises yield criteria ($\|\boldsymbol{\tau}\|/p \leq \mu(I)$) and a flow rule ($\mathbf{D} \propto \boldsymbol{\tau}$), both standard ingredients of plasticity models for quasi-static granular flows (Goddard 2014). The dependence of μ upon I is, however, particular of rapid and dense granular flows. The following empirical relationship was proposed

$$\mu = \mu_s + \frac{\Delta\mu}{I_0/I + 1}, \quad (2.2)$$

where μ_s is a “static” friction coefficient, $\Delta\mu \approx 0.3$ and $I_0 \approx 0.3$ are empirical parameters (GDR MiDi 2004; da Cruz *et al.* 2005; Jop *et al.* 2006).

Taking now into account the weak compressibility of granular flows, the strain rate

tensor needs to be rewritten as the sum of an isotropic and a deviatoric part:

$$D_{ij} = \frac{1}{n} \text{tr}(\mathbf{D}) \delta_{ij} + (D_{ij} - \frac{1}{n} \text{tr}(\mathbf{D}) \delta_{ij}) = \frac{1}{n} \text{tr}(\mathbf{D}) \delta_{ij} + S_{ij}, \quad (2.3)$$

where n is the number of space dimensions and S_{ij} corresponds to the isochoric part of the deformations. As in plasticity models, stress is related to strain rate by the combination of a flow rule and a yield criterion. We assume that it takes the simple form

$$\frac{\tau_{ij}}{p} = \mu(I) \frac{S_{ij}}{\|\mathbf{S}\|} + \frac{\mu_b}{n} \frac{\text{tr}(\mathbf{D}) \delta_{ij}}{\|\mathbf{S}\|}, \quad (2.4)$$

where we introduced an additional bulk friction coefficient μ_b arising from isotropic compression or dilation of the granular flow. This form is rather common in plasticity (see for instance Krishnaraj & Nott (2016)) but, to our knowledge, has not been proposed yet to model dense granular flows (Trulsson *et al.* (2013) used a similar form but with $\|\mathbf{D}\|$ instead of $\|\mathbf{S}\|$ while Börzsönyi *et al.* (2009) used $\|\mathbf{S}\|$ but did not introduce a bulk friction coefficient). The bulk friction coefficient may also depend on I but to our knowledge, such dependence has not been reported yet. We thus assume μ_b to be a constant in the following. The inertial number is also rewritten as

$$I = \frac{2\|\mathbf{S}\|d}{\sqrt{p/\rho}}. \quad (2.5)$$

It is worth pointing that the volume fraction ϕ appears implicitly in the definition of the inertial number, since we have $\rho = \varrho_s \phi$, with ϱ_s the density of grains. At equal confinement pressure, the more dilute the flow, the higher the forces transmitted through the granular skeleton. These forces scale as $d^2 p / \phi$, implying a dependence of the microscopic rearrangement time scale upon volume fraction.

The granular flow is envisioned as a continuum and homogeneous compressible medium governed by the Navier-Stokes equations. In dimensionless form, they read

$$\partial_t \phi + \partial_i (\phi u_i) = 0 \quad (2.6)$$

$$R^2 \phi (\partial_t u_i + u_j \partial_j u_i) = -\partial_j p + \partial_j \tau_{ij} - G \phi \vec{z}, \quad (2.7)$$

where u_i is a dimensionless component of the velocity vector, $R^2 = \varrho \Phi U^2 / P$ is an effective Reynolds number and $G = \varrho g \Phi L / P$ a dimensionless gravity number ($\varrho, g, \Phi, U, P, L$ are respectively the grain density, the gravitational acceleration, a unit volume fraction, a unit velocity, a unit pressure and a unit length). τ_{ij} is given by Eq. (2.4) and the system is closed by providing an equation of state linking the volume fraction ϕ to other variables.

For quasi-static granular flows, plasticity models usually provide a relationship between p and ϕ through the critical state concept (Rao & Nott 2008). In contrast, experiments and DEM simulations suggest that the volume fraction of dense inertial granular flows is inversely proportional to the inertial number I (GDR MiDi 2004; da Cruz *et al.* 2005; Börzsönyi *et al.* 2009), reflecting the propensity of grains to escape regions of high shear. Based on this observation, we assume here that ϕ is also a sole function I . The specific form of the functional is not necessary here. In agreement with numerical simulations, we simply require that the derivative $\phi' = \partial_I \phi$ be negative. Given this scaling, the mass conservation equation transforms into

$$\partial_t I + u_i \partial_i I = -\frac{\phi}{\phi'} \partial_i u_i. \quad (2.8)$$

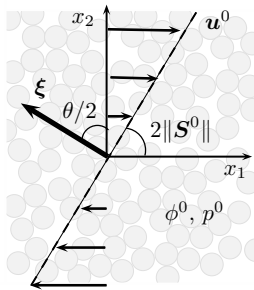


FIGURE 1. Unidirectional shear in the plane (x_1, x_2) . \mathbf{u}^0 and $2\|\mathbf{S}^0\|$ are the flow velocity profile and shear rate. ϕ^0 and p^0 are the volume fraction and granular pressure. $\boldsymbol{\xi}$ is the perturbation wavevector and $\theta/2$ the inclination angle with respect to the axis x_2 .

3. Base flow and perturbation

We consider the case of a simple plane shear flow in two dimensions ($\mathbf{x} = (x_1, x_2)$) in absence of gravity (Fig. 1). The flow is assumed to follow the compressible rheology described above. The principal shear direction is along x_1 so that the velocity vector has an only non-zero component $\mathbf{u} = (u_1, 0)$. The strain rate matrix has thus a unique non-zero component $D_{12} = \partial_2 u_1 / 2$. Plane shear is an isochoric deformation ($\text{tr} \mathbf{D} = 0$) so that $\mathbf{D} = \mathbf{S}$. Conservation of momentum implies that the pressure and the shear stress are uniform through the flow, yielding a uniform friction coefficient μ , a uniform inertial number and a uniform volume fraction. By definition of I , \mathbf{S} is also uniform so that u_1 is a linear function of x_2 (Fig. 1).

Mathematically, a problem is said to be well-posed if the growth rate of any small perturbation is bounded. To check the existence of such bound, it is in general sufficient to consider small perturbations in the limit of infinitely large wavenumbers (small wavelengths), limit at which the base flow appears locally uniform (Barker *et al.* 2015). It is thus legitimate to look for small perturbations over the base shear flow of the form

$$\mathbf{u} = \mathbf{u}^0 + \tilde{\mathbf{u}} e^{i\boldsymbol{\xi} \cdot \mathbf{x} + \lambda t}, \quad |\boldsymbol{\xi}| \rightarrow \infty \quad (3.1)$$

where $\boldsymbol{\xi} = (\xi_1, \xi_2)$ is the wave vector ($|\bullet|$ is the euclidean norm), λ is the growth rate and $\tilde{\mathbf{u}}$ the amplitude of the perturbation. The scaling of λ at large $|\boldsymbol{\xi}|$ then governs the well or ill-posed behaviour of the equations. Three cases may be distinguished as $|\boldsymbol{\xi}| \rightarrow \infty$:

- if $\lambda \rightarrow +\infty$, the system is unstable and small perturbations grow at an unbounded rate, suggesting ill-posedness,
- if $0 < \lambda < \infty$, the system is unstable but growth rates are bounded so that the problem remains well-posed,
- if $\lambda < 0$, the system is stable and thus the problem is well-posed.

$$(\mathbf{D}, \mathbf{S}, \|\mathbf{S}\|, p, I, \mu, \phi) = (\mathbf{D}^0, \mathbf{S}^0, \|\mathbf{S}^0\|, p^0, I^0, \mu^0, \phi^0) + (\tilde{\mathbf{D}}, \tilde{\mathbf{S}}, \|\tilde{\mathbf{S}}\|, \tilde{p}, \tilde{I}, \tilde{\mu}, \tilde{\phi}) e^{i\boldsymbol{\xi} \cdot \mathbf{x} + \lambda t},$$

where $\tilde{D}_{ij} = \tilde{S}_{ij} + \text{tr}(\tilde{\mathbf{D}})\delta_{ij}$, $\|\tilde{\mathbf{S}}\| = 2\|\mathbf{S}^0\|\|\tilde{S}_{12}\|$ and the components of $\tilde{\mathbf{S}}$ are given in appendix A. In addition, since μ and ϕ are considered sole functions of I , we write $\tilde{\mu} = \mu' \tilde{I}$ and $\tilde{\phi} = \phi' \tilde{I}$, where the primes stand for a derivative with respect to I^0 . By definition, the perturbed inertial number may be written at first order

$$\frac{\tilde{I}}{I^0} = \frac{\|\tilde{\mathbf{S}}\|}{\|\mathbf{S}^0\|} - \frac{\tilde{p}}{2p^0} + \frac{\phi' \tilde{I}}{2\phi^0}, \quad (3.2)$$

Equations	$\tilde{\mathbf{u}}$	\tilde{I}
Mass	$O(\xi)$	$O(\xi)$
Momentum	$O(\xi^2)$	$O(\xi)$

TABLE 1. Order of the leading order terms in Navier-Stokes equations depending on the variable and the equation.

which allows eliminating \tilde{p} in the equations. Given the rheological law (2.4), the stresses are, to first order,

$$\tilde{\tau}_{ij} = \frac{p^0}{\|\mathbf{S}^0\|} \left(\mu^0 \tilde{S}_{ij} + \frac{\mu_b}{n} \text{tr}(\tilde{\mathbf{D}}) \delta_{ij} \right) \quad \text{for } i = j, \quad (3.3)$$

$$\tilde{\tau}_{12} = \tilde{\tau}_{21} = \mu^0 \tilde{p} + \mu' p^0 \tilde{I}. \quad (3.4)$$

By choosing appropriately the scales L , U , Φ , and P we can always normalize the base flow so that $\|\mathbf{S}^0\| = 1/2$, $\phi^0 = 1$ and $p^0 = 1$. Inserting the previous developments in the Navier-Stokes equations, and keeping only linear contributions and leading order terms in the high wavenumber limit (according to table 1) leads to an eigenvalue problem of the form $(\mathbf{A} - \lambda \mathbf{B})\tilde{\mathbf{y}} = 0$ with $\tilde{\mathbf{y}} = (\tilde{\mathbf{u}}, \tilde{I})^T$ and \mathbf{A} and \mathbf{B} are 3×3 matrices given in appendix B. Non-trivial solutions to this system are obtained if $\det(\mathbf{A} - \lambda \mathbf{B}) = 0$, which simplifies to

$$a_3 \lambda^3 + a_2 \lambda^2 + a_1 \lambda + a_0 = 0, \quad (3.5)$$

with

$$\begin{aligned} a_3 &= \phi' R^4, \\ a_2 &= \frac{2}{n} \phi' R^2 |\boldsymbol{\xi}|^2 (\mu_b + (2n-1)\mu - n \sin \theta), \\ a_1 &= -R^2 |\boldsymbol{\xi}|^2 \beta (1 - (\mu - r) \sin \theta) \\ &\quad + \frac{1}{n} \phi' |\boldsymbol{\xi}|^4 \mu (4(\mu_b + (n-1)\mu) - 2n \sin \theta - (2\mu_b + (n-2)\mu) \sin^2 \theta), \\ a_0 &= |\boldsymbol{\xi}|^4 \beta (-2r + \mu(\mu - r) \sin \theta - (\mu - 2r) \sin^2 \theta), \end{aligned}$$

where $\theta = 2 \tan^{-1}(\xi_1/\xi_2)$, $r = \mu' I / (2 - \phi' I)$, $\beta = 2/I - \phi'$ and $n = 2$ or 3 dimensions. To simplify notations, we dismissed the superscript over all base flow variables.

4. Growth rate in the high wavenumber limit

Analysis of Eq. (3.5) shows that two different scaling arises for λ at large $|\boldsymbol{\xi}|$: $\lambda \propto O(1)$ and $\lambda \propto O(|\boldsymbol{\xi}|^2)$. Both cases are considered analytically in the following.

4.1. $\lambda \propto O(|\boldsymbol{\xi}|^2)$

Writing $\lambda = \lambda' |\boldsymbol{\xi}|^2$, with $\lambda' \propto O(1)$, we retain only the $O(|\boldsymbol{\xi}|^6)$ terms in (3.5) and get the quadratic equation

$$n \lambda'^2 + 2aR^{-2} \lambda' + bR^{-4} = 0, \quad (4.1)$$

with

$$\begin{aligned} a &= \mu_b + (2n-1)\mu - n \sin \theta, \\ b &= \mu(4(\mu_b + (n-1)\mu) - 2n \sin \theta - (2\mu_b + (n-2)\mu) \sin^2 \theta). \end{aligned}$$

This yields two solutions for λ :

$$\lambda_{1,2} = -\frac{|\xi|^2 a}{nR^2} \left(1 \pm \sqrt{\Delta}\right), \quad \Delta = 1 - \frac{nb}{2a^2}. \quad (4.2)$$

When $b > 0$, $\Delta < 1$ so that $\text{Re}(\lambda_{1,2}) < 0$ and thus the growth rate is always negative. When $b < 0$, then $\Delta > 1$ and one of the roots become positive with a growth rate scaling as $|\xi|^2$, indicating ill-posedness behaviour of the equations. Taking $X = 1/\sin \theta$, $b > 0$ is equivalent to

$$4(\mu_b + (n-1)\mu)X^2 - 2nX - 2\mu_b - (n-2)\mu > 0, \quad |X| \geq 1. \quad (4.3)$$

Since the determinant of this quadratic equation is always positive, roots are pure reals and they must lie in the interval $] -1, 1[$ for b to be always positive. This gives a lower bound $\mu_b > n - (3n-2)\mu/2$ under which b becomes negative and the rheology becomes ill-posed. This yields in 2D, $\mu_b < 2-2\mu$ and in 3D, $\mu_b < 3-7\mu/2$ (Fig. 2(a)). The ill-posed direction is found where b takes a minimum, that is for angles with x_2 of $\theta/2 = \pi/4$.

4.2. $\lambda \propto O(1)$

In this case, the growth rate $\text{Re}(\lambda)$ is always bounded for large $|\xi|$, so that this particular solution does not lead to ill-posedness of the problem. To see that, as $\lambda \propto O(1)$, we retain only the $O(|\xi|^4)$ terms and Eq. (3.5) simplifies to

$$\lambda_3 = \frac{3\beta}{|\phi'|b} (-2r + \mu(\mu-r)\sin\theta - (\mu-2r)\sin^2\theta), \quad (4.4)$$

which proves the existence of an upper bound. In other words, if λ_3 becomes positive, the flow becomes unstable in the high wavenumber limit but the problem remain well-posed. For $b > 0$ ($b < 0$ was proven to generate ill-posedness), the sign of λ_3 is determined by the sign of the nominator. Posing $X = 1/\sin \theta$, the latter is negative if

$$-2rX^2 + \mu(\mu-r)X - (\mu-2r) < 0, \quad |X| > 1. \quad (4.5)$$

This condition is verified if (i) the determinant $\Delta = \mu^2(\mu-r)^2 - 8r(\mu-r)$ is negative or (ii) the determinant is positive, but the real roots lie inside the interval $[-1, 1]$, that is if $\sqrt{\Delta} < 2r - \mu(\mu-r)$. Thus, λ_3 is negative if

$$\mu^2(\mu-r)^2 - 8r(\mu-r) < \max(0, 2r - \mu(\mu-r))^2. \quad (4.6)$$

It is easy to show that $0 < r < \Delta\mu/8$ and thus, for usual rheology parameters ($\Delta\mu \approx 0.26$, $\mu_s \approx 0.38$), $2r - \mu(\mu-r)$ is always negative and the stability condition simplifies to

$$\mu^2(\mu-r)^2 - 8r(\mu-r) < 0, \quad \text{with } r = \frac{\mu'I}{2 + \phi'I}. \quad (4.7)$$

The stability domain is shown in Fig. 2(b) in the plane $(I/I_0, \Delta\mu)$. Its extent shrinks when increasing compressibility.

4.3. Summary

To summarize, the compressible $\mu(I)$ -rheology leads to plane shear flows that are unstable outside of the domain defined by Eq. 4.7 (Fig. 2). However, if compressibility is associated with sufficient dissipation (i.e., $\mu_b > n - (3n-2)\mu/2$) the growth rate of the unstable modes remains bounded in the high wavenumber limit, and the rheology remains well-posed at any inertial number.

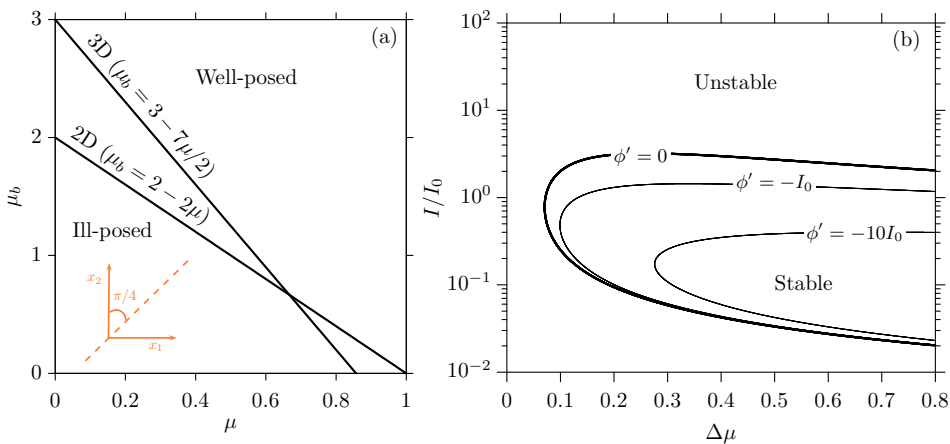


FIGURE 2. (a) Well-posed and ill-posed domains in the $\mu - \mu_b$ plane for $n = 2$ and $n = 3$ dimensions. The ill-posed direction is indicated in orange. (b) Stability domain of the compressible $\mu(I)$ -rheology in the $I/I_0 - \Delta\mu$ plane with various compressibility ϕ' . Other parameters are $\mu_b = 2$, $\mu_s = 0.383$.

It is instructive to note that, in the limit of an incompressible media (putting $\phi' = 0$), $r = \mu'I$ and the condition for stability becomes

$$\mu^2(\mu - \mu'I/2)^2 - 4\mu'I(\mu - \mu'I) < 0, \quad (4.8)$$

an expression equivalent to the well-posedness domain defined by Barker *et al.* (2015). It is indeed straightforward to see that, putting $\phi' = 0$, the growth rate $\lambda = -a_0/a_1$ is now scaling as $|\xi|^2$ so that, as soon as the instability appears, the incompressible rheology becomes ill-posed.

Another interesting limit is obtained putting $\mu' = 0$, which yields the classical Coulomb constant friction model. In this case, it is immediate to see that (4.7) is never verified for $\mu > 0$ so that the shear flows described by a Coulomb rheology are always unstable but remain well-posed since the condition $\mu_b > n - (3n - 2)\mu/2$ is not affected by the value of μ' .

Let us infer *a posteriori* the impact of embracing the volume fraction variations in the definition of the inertial number. Ignoring these, the last term in (3.1) is dismissed, leading to a dispersion relation similar to Eq. (3.5), but with $\beta = 2/I$ and $r = \mu'I/2$, both independent of compressibility. As a result, the stability domain (4.7) would now be independent of ϕ' . The well-posedness criteria is however not affected by ignoring the inertial number dependence on volume fraction.

5. Conclusion

In this communication we showed that compressibility generally regularizes the $\mu(I)$ -rheology, proposed for dense inertial granular flows. Volume changes need to be associated with a sufficient dissipation to make the problem well-posed at all inertial numbers. This condition is expressed in terms of an inequality between μ_b , the bulk friction coefficient associated with volume changes and μ the friction coefficient associated with shear: $\mu_b > 3 - 7\mu/3$ in 3 dimensions and $\mu_b > 2 - 2\mu$ in 2 dimensions. The direction at which

the problem becomes ill-posed is always making an angle $\pi/4$ with the principal shear direction.

The compressible $\mu(I)$ -rheology was also shown to be unstable for a large range of flow conditions, both at large and small inertial number, in the high wavenumber limit. Secondary flows and resonances of dense inertial granular flows may be explained by such instabilities (Börzsönyi *et al.* 2009; Trulsson *et al.* 2013; Brodu *et al.* 2015; Krishnaraj & Nott 2016). Our analysis is limited to high wavenumbers and thus cannot provide an estimation of the most unstable wavelength and its associated growth rate. To get this information, the full dispersion equation needs to be solved via numerical methods (Schmid & Henningson 2001).

Finally, it is legitimate to wonder if any hypothetical variations of μ_b with I or other flow variables would modify significantly the preceding results. In the case of a pure shear base flow, the contributions arising from the variations of μ_b would not appear in the linearised equations so that our conclusions are still valid. Additional experimental and numerical work is thus needed to quantify the bulk friction associated to volume changes in dense granular flows.

More generally, this study shows that the generation and propagation of acoustic waves inside granular flows are likely to play a crucial role in their flowing characteristics. Depending on their geometry and their elastic properties, boundaries may either absorb or restitute part of the acoustic energy to the flow and produce apparent non-local effects.

Appendix A. Perturbed strain rate

$$\tilde{\mathbf{D}} = \mathbf{i} \begin{pmatrix} \xi_1 \tilde{u}_1 & \frac{1}{2}(\xi_2 \tilde{u}_1 + \xi_1 \tilde{u}_2) \\ \frac{1}{2}(\xi_2 \tilde{u}_1 + \xi_1 \tilde{u}_2) & \xi_2 \tilde{u}_2 \end{pmatrix}, \quad (\text{A } 1)$$

$$\tilde{\mathbf{S}} = \mathbf{i} \begin{pmatrix} \frac{1}{n}(2\xi_1 \tilde{u}_1 - \xi_2 \tilde{u}_2) & \frac{1}{2}(\xi_2 \tilde{u}_1 + \xi_1 \tilde{u}_2) \\ \frac{1}{2}(\xi_2 \tilde{u}_1 + \xi_1 \tilde{u}_2) & \frac{1}{n}(\xi_2 \tilde{u}_2 - 2\xi_1 \tilde{u}_1) \end{pmatrix}. \quad (\text{A } 2)$$

Appendix B. Eigenvalue problem

The eigenvalue problem in the high wavenumber limits reads $(\mathbf{A} - \lambda \mathbf{B})\tilde{\mathbf{y}} = 0$ with

$$\mathbf{A} = \begin{pmatrix} 2\xi_1^2\alpha + 2\mu\xi_2^2 - 2\xi_1\xi_2 & 2\xi_1\xi_2\alpha - 2\xi_1^2 & i\xi_2(\beta\mu - \mu') - i\beta\xi_1 \\ 2\xi_1\xi_2\alpha - 2\xi_2^2 & 2\xi_2^2\alpha + 2\mu\xi_1^2 - 2\xi_1\xi_2 & i\xi_1(\beta\mu - \mu') - i\beta\xi_2 \\ i\xi_1 & i\xi_2 & 0 \end{pmatrix}, \quad (\text{B } 1)$$

$$\mathbf{B} = - \begin{pmatrix} R^2 & 0 & 0 \\ 0 & R^2 & 0 \\ 0 & 0 & \phi' \end{pmatrix}, \quad (\text{B } 2)$$

where $\alpha = \mu_b/n + (1 - 1/n)\mu$ and $\beta = 2/I - \phi'$.

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